The Thue-Morse Sequence and Boolean Algebra in Music Composition

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1 The Thue-Morse Sequence

The Thue-Morse sequence is created by stringing together 1's and 0's following three simple steps:

1. Start with 0.
2. The next segment of the sequence will be the Boolean complement of everything that comes before it.
3. Repeat step 2.

To demonstrate how this is built:

0
0 1
0 1 1 0
0 1 1 0 1 0 0 1
0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0

What I find to be particularly compelling about the Thue-Morse sequence is that it operates by its own rules at different levels of magnitude. Let's call the unit "0 1 1 0," unit **A**, and the unit "1 0 0 1," unit **B**. Notice then that

0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0

is

0 1 1 0   1 0 0 1   1 0 0 1   0 1 1 0

is

A B B A ,

and if we continue the sequence longer,

0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0

is

A B B A   B A A B   B A A B   A B B A , which is identical to 0 1 1 0   1 0 0 1   1 0 0 1   0 1 1 0 .

So we can see that the pattern at the most microscopic level mirrors the pattern at the most macroscopic level as well. (This is rigorously provable by induction, noting that because we can put each level into the same form, it follows that each subsequent level will continue to function the same way as the last.)

2 Applying the sequence to music

In this section, we will consider three applications of the Thue-Morse Sequence to music. The second and third applications show up in an etude I wrote in conjunction with this mathematical exploration, which can be found at the end of this paper. The first application, I have not used in my music yet. However, I found it to be a productive and thought-provoking starting-point.

Application 1 When I began thinking about writing music that would incorporate the Thue-Morse Sequence, my first thought was of applying the binary numbers to intervals. That is because the most familiar role of Arabic numerals in music theory is their application in denoting intervals and scale degrees.

In typical Western classical scales, two consecutive notes can be either a half step or a whole step apart. This is a binary situation, so it makes sense to apply a binary sequence to it. Furthermore, there is a long history of European composers creating new scales by following *symmetric patterns* of half and whole steps. (The most overwhelmingly common scales in Western classical music follow non-symmetrical patterns of half and whole steps.)

So, let's build a scale with intervals following the pattern of the Thue-Morse sequence, with 0's translating into half steps and 1's translating into whole steps. We'll start all our examples arbitrarily on G, for consistency and ease of comparison. (I'm fairly sure I chose G because it's the lowest note of my native instrument, the violin. If you like a different number, or want an extra music theory challenge, feel free to follow along constructing your own examples!)

$$\mathrm{G}_{+0}\ \mathrm{A}\flat_{+1}\ \mathrm{B}\flat_{+1}\ \mathrm{C}_{+0}\ \mathrm{C}\sharp_{+1}\ \mathrm{D}\sharp_{+0}\ \mathrm{E}_{+0}\ \mathrm{F}_{+1}\ \mathrm{G}$$

For anyone steeped in Western classical music theory, you'll notice that at this point we've completed an octave. But remember that next we have the Boolean complement of the sequence so far, so if we keep going with the scale the next octave of the scale is actually going to be different! What we have here is super cool (in my opinion!) because it's a scale that doesn't repeat at the octave. The idea of scales that don't repeat at the octave, which can take on many different manifestations, is found most frequently in microtonal compositions, such as those of Wendy Carlos, rather than in the twelve-tone universe we're working in here by–admittedly unfortunate–default.

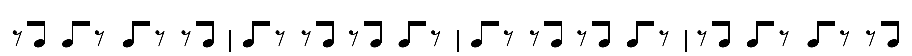
So this scale is going to be more than an octave if we let it. (We could also have cut it off at just five notes or six–a scale is just a collection of however many notes you let it be.) Let's see how long we can go before it starts to repeat:

$$\begin{array}{l}
\mathrm{G}_{+0}\ \mathrm{G}\sharp_{+1}\ \mathrm{A}\sharp_{+1}\ \mathrm{C}_{+0}\ \mathrm{C}\sharp_{+1}\ \mathrm{D}\sharp_{+0}\ \mathrm{E}_{+0}\ \mathrm{F}_{+1}\\
\mathrm{G}_{+1}\ \mathrm{A}_{+0}\ \mathrm{A}\sharp_{+0}\ \mathrm{B}_{+1}\ \mathrm{C}\sharp_{+0}\ \mathrm{D}_{+1}\ \mathrm{E}_{+1}\ \mathrm{F}\sharp_{+0}\\
\mathrm{G}_{+1}\ \mathrm{A}_{+0}\ \mathrm{A}\sharp_{+0}\ \mathrm{B}_{+1}\ \mathrm{C}\sharp_{+0}\ \mathrm{D}_{+1}\ \mathrm{E}_{+1}\ \mathrm{F}\sharp_{+0}\\
\mathrm{G}_{+0}\ \mathrm{G}\sharp_{+1}\ \mathrm{A}\sharp_{+1}\ \mathrm{C}_{+0}\ \mathrm{C}\sharp_{+1}\ \mathrm{D}\sharp_{+0}\ \mathrm{E}_{+0}\ \mathrm{F}_{+1}\\
\mathrm{G}_{+1}\ \mathrm{A}_{+0}\ \mathrm{A}\sharp_{+0}\ \mathrm{B}_{+1}\ \mathrm{C}\sharp_{+0}\ \mathrm{D}_{+1}\ \mathrm{E}_{+1}\ \mathrm{F}\sharp_{+0}\\
\mathrm{G}_{+0}\ \mathrm{G}\sharp_{+1}\ \mathrm{A}\sharp_{+1}\ \mathrm{C}_{+0}\ \mathrm{C}\sharp_{+1}\ \mathrm{D}\sharp_{+0}\ \mathrm{E}_{+0}\ \mathrm{F}_{+1}\\
\mathrm{G}_{+0}\ \mathrm{G}\sharp_{+1}\ \mathrm{A}\sharp_{+1}\ \mathrm{C}_{+0}\ \mathrm{C}\sharp_{+1}\ \mathrm{D}\sharp_{+0}\ \mathrm{E}_{+0}\ \mathrm{F}_{+1}\\
\mathrm{G}_{+1}\ \mathrm{A}_{+0}\ \mathrm{A}\sharp_{+0}\ \mathrm{B}_{+1}\ \mathrm{C}\sharp_{+0}\ \mathrm{D}_{+1}\ \mathrm{E}_{+1}\ \mathrm{F}\sharp_{+0}
\end{array}$$

And we could keep going but I think we're starting to see what happens. We've got our familiar **ABBABAAB...** pattern unfolding, this time with each octave of the scale being either of pattern **A** or of pattern **B**. What's compelling about this, to me, is that we never get to a point where the pattern will just straight-forwardly repeat, because we're always going to be dealing with the Thue-Morse Sequence even at the highest level, which never turns into a full repetition of itself because it is always adding on the opposite of itself thus far.

Application 2 If we forget about 1's and 0's and take the approach that the Thue-Morse Sequence can be applied to any binary, any either-or situation, then a second immediately logical connection to music is the binary of sound and silence.

To put this in more Boolean terms, we can say that if we have a regular pulse, on each beat of that pulse either there is an attack (a sound) or there isn't. If we translate 0 as a rest and 1 as an attack, with a regular pulse of eighth notes in 4/4 time for something simple, we get this rhythm:



Unfolding here we have two distinct rhythmic patterns a measure can have, either

A 

...written more concisely as 

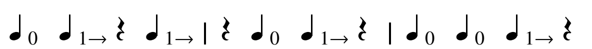
or

B 

...written more concisely as  .

Then the very familiar-by-now pattern **A B B A B A A B B A A B A B B A . . .** guides the layout of these two measures/rhythmic patterns into a longer form.

Application 2.5 In the two-instrument etude attached to the end of this paper, I use both the rhythmic construction shown above (in the upper voice) and a second rhythmic interpretation of the Thue-Morse Sequence, where 0's and 1's translate into how many rests follow an attack, either no rest or one rest. For this second interpretation, which makes the rhythm of the lower voice, the regular pulse is the quarter note rather than the eighth note. That fundamentally different but equally justifiable rhythmic interpretation of the sequence looks like this:



What I love about juxtaposing this rhythm with the first rhythm is that they have two different periodicities. The first rhythm cycles through **A B B A** at the levels of half a measure (the most micro level), one measure, two measures, four measures and so on. In contrast, the second rhythm cycles through **A B B A** at the levels of *one-and-a-half* measures (its most micro level), three measures, six measures, twelve measures and so on. Layering these two rhythms on top of each other creates more rhythmic variety because of the contrasting periodicities.

Application 3 While I found the scale/interval pattern that was my initial instinct compelling, I wanted to explore a more definite way of using the sequence to *strictly build a melody*. This led me to consider a third familiar case in Western classical music theory that can be considered a Boolean situation.

The standard way intervals are taught in this tradition of musicmaking is in pairs, every interval having an *inverse*. The inverse of an interval is, by common definition, the interval you need to add to it to reach an octave. For instance, from G, if we leap up a Major 3rd, then we are at the pitch B. If we want to leap up again and get back to a G, we have to leap up a minor 6th since the interval from B up to G is a minor 6th. Therefore, we say the *inverse* of a Major 3rd is a minor 6th.

I tried out a Thue-Morse-derived melody where 0 stood for a given interval and 1 stood for its inverse. I quickly decided that all such melodies were useless–or, at least, of little interest to me–because every other note almost would be a repetition of the starting note, since the combination of 0 then 1, by definition of inverse intervals, transports the pitch right back to the pitch it started at, just an octave displaced, and vice versa (vice versa meaning the combination of 1 then 0 does the same, by commutivity of addition). The reader can test this outcome as an exercise. So many octaves, and only two other pitches ever interceding, made a boring melody in my estimation.

So, to make something marginally more exciting, I took the principle of inverse intervals and resized it. What if the intervals don't add up to an octave? Since the octave was my problem, I'd simply replace it with something else. (Because of acoustics and Western classical music training, it's really only octaves that will get stuck in the particular way I was dissatisfied with, but there really isn't much to be gained from going into detail on the reasons for that in this short paper.) It could have been any other interval, but for my purposes I chose a Perfect 5th.

There are three pairs of intervals which add up to a Perfect 5th: minor 2nd and tritone; Major 2nd and Perfect 4th; minor 3rd and Major 3rd. I didn't even consider the minor 2nd and tritone pairing. A tritone is its own inverse, that is, two tritones add up to an octave, so I saw in the prospect of tritones that same octave monotony arising. In addition, tritones have a very particular sound which is hard to get away from or hear in more than one way. The minor 3rd-Major 3rd pairing also struck me as underwhelming for a melody, because arpeggios made up of alternating minor and Major 3rds are a chordal/accompanimental staple of Western classical music, which made me less than excited about their potential for making a stirring melody. All this is more or less arbitrary, though. Another composer might have jumped on one of these two pairings with enthusiasm.

As for me, I jumped on Major 2nds and Perfect 4ths. The first page of music you'll see at the end of this paper is a melody constructed purely by following, uninterrupted and unaltered, the Application 2 rhythm and the Thue-Morse Sequence manifested as a melody of Major 2nds and Perfect 4ths. In the two-instrument etude, I did use the minor 3rd-Major 3rd version of this melodic construction (less strictly). I found that I loved it as a *counterpoint* to the first voice, perhaps precisely because of the chordal/harmonic implications of the endlessly spinning minor and Major 3rds.

I apply the Thue-Morse Sequence to these interval pairs in the same way as the scale (Application 1). Start with a given pitch, in this case G, then the next pitch is a Perfect 4th up from G (C), the next a Major 2nd up from C (D), the next another Major 2nd up (E), the next a Perfect 4th up from that one (A), and so on. If the reader would like to try building this melody out to the point that it repeats, before looking at my music, I encourage you with the sense of wonder I felt as I watched surprising detours of *non-repetition* unfold under my pen at various points along the way. (That's what's so cool about writing with the Thue-Morse Sequence!)

3 Ways to continue thinking about Boolean algebra and music

As we've seen in the preceding three-and-a-half applications of the Thue-Morse Sequence to music, there are many ways–and a true *variety* of ways–to consider musical decisions as binary situations. Ascending a half step or a whole step; playing a note or not playing a note; resting or not resting between two notes; leaping by one interval or its "inverse." And there are all manner of ways to build more complex ideas out of these simple ones, whether by combining two or more, as I have done in my etude–and there were other ways I could have combined my elements which would have wrought completely different results–or by expanding, contracting, nudging, turning sideways. There is no single correct way to apply any of this patterning. Anything that can be described as a binary operation, limited or simplified to a choice between two things, can have a Boolean algebraic pattern or operation applied to it.

As we saw in this paper, I decided fairly arbitrarily that pairs of intervals which add up to a Perfect 5th could be Boolean binaries. By limiting myself to writing using only two intervals, I could then easily say, "Well, it's either one or the other, so this is a situation where any Boolean logic can apply."

The beauty of exploring math in the arts is that the looser constraints of a discipline like music opens up these more malleable areas of creativity. While trying to solve a mathematical equation, I usually do not think to make arbitrary limitations, especially because they might slow down my problem-solving or limit the applicability of my results. If I am trying to prove a theorem on the Natural Numbers, I am unlikely to see an immediate benefit in trying to prove the theorem only for the numbers 2 and 5–and I don't imagine I would even think of trying to do that to begin with, because I have a task to accomplish that requires me to generalize across *all* the Natural Numbers. I am only going to think to narrow my focus with my Natural Numbers problem to the extent that it carries me closer to solving for something general.

Music inhabits an interesting threshold for me. On the one hand, music is an abstract artform almost as a rule. On the other hand, it is not theoretical, but is applied, in the end is the engineering and production of intentional sound. In my experience, this means that music has both a natural, logical relationship to theoretical math, as opposed to applied math, but that it also has a manifestation in the *particular*, the *specific*, which is not so very related to theoretical math. Thus, when we explore mathematical ideas in the context of music, we are encouraged to marry the theoretical and the practical in a distinct, unusual and uncommonly generative way.

I affirm that I have adhered to the Honor Code in this assignment. RMA

Thue-Morse Etude - basis

Rania Adamczyk

Thue-Morse Etude for Two

Rania Adamczyk


21
Fl.
Bsn.
mf
pp
f
26
p
sub p
fp
31
ppp
mp
36
41